

# (A note on) self-similarity in granular media

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It is shown that the phenomenon of self-similarity appears in granular media, with an intergrain potential  $V \propto \delta^{p+1}$ ,  $p > 1$ , where  $\delta$  is the overlap between the grains. Although this fact can be traced back in the literature, this has not been put explicitly.

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Granular materials are nonlinear media which are of practical interest for many applications. They are of interest also from the theoretical point of view: in one-dimensional systems it has been shown by theoretical [1, 2] as well as experimental [3] research, the existence of localised travelling wave solutions. In practice, these solitary waves are generated by impacting one end of a chain of grains. Despite the large amount of recent work on the subject [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], the physics of granular media remains a challenge.

Scaling laws have wide applications in science and engineering [18, 19, 20]. They give evidence of the property of self-similarity of phenomena, that is of the fact that they reproduce themselves after a rescaling of some variables and/or parameters. Self-similarity has been used in the past in order to transform systems of partial differential (PDEs) equations into systems of ordinary differential equations, with the hope that the initial problem will get easier to solve. This is one of the standard methods for obtaining exact solutions of PDEs. Nowadays, the search for exact solutions is focused to understand the mathematical structure of the solutions and, hence, to get a deeper understanding of the physical phenomena described by them.

In this Letter, it is shown that the phenomenon of self-similarity appears in one-dimensional granular media with intergrain potentials  $V \propto \delta^{p+1}$ ,  $p > 1$ , where  $\delta$  is the overlap between the grains, in case there is no pre-compression.

To illustrate the phenomenon of self-similarity in granular media, let us consider the scattering of solitary waves in a one-dimensional linear chain interacting with a potential as the one mentioned previously and with the form of a granular container [21, 22], as shown in Fig. 1.

We shall assume that the granular container has a total of  $M$  beads. There are two set of beads with  $N_1$  beads located on the lhs,  $N_2$  on the rhs, both sets have beads with radii  $a$ . Between them there are  $L$  beads with radii  $b$  ( $a > b$ ).

Let  $x_i(t)$  be the displacement of the center of the  $i$ -th grain from its initial equilibrium position, and assume that the  $i$ -th grain, of mass  $m_i$ , has neighbors of different radii and/or mechanical properties. Then, in absence of load and in a frictionless medium, the equation of motion

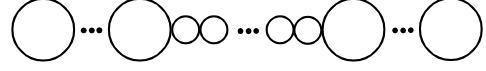


FIG. 1: Schematic granular container used in the calculations.

for the  $i$ -th sphere reads

$$m_i \frac{d^2 x_i}{dt^2} = k_1 (x_{i-1} - x_i)^p - k_2 (x_i - x_{i+1})^p, \quad (1)$$

where it is understood that the brackets take the argument value if they are positive and zero otherwise, ensuring that the spheres interact only when in contact.

In order to have realistic results, we shall assume that the system consists of stainless-steel beads (see [3] for their properties), with radii  $a = 4$  mm and  $b = 2$  mm. The number of beads is  $N_1 = 20$ ,  $N_2 = 20$  and  $L = 100$ . We also choose  $\beta = 10^{-5}$  m,  $2.36 \times 10^{-5}$  kg and  $\alpha = 1.0102 \times 10^{-3}$  s as units of distance, mass and time, respectively. Through out the paper we assume that initially all beads are at rest, that is,

$$\begin{aligned} u_i(0) &= 0, \quad i = 1, \dots, M \\ \dot{u}_i(0) &= 0, \quad i = 2, \dots, M, \end{aligned}$$

except for the first bead at the left side of the chain. This bead is supposed to have a nonzero value of velocity  $\dot{u}_1(0) = v_0$  in order to generate the soliton-like perturbation in the chain. A simple analysis of the behavior of the granular system under rescalings of the impact velocity from  $v_0$  to  $\lambda v_0$ , shows that the solutions of the set of ODEs is self-similar. That is, it is found that

$$X(t; \lambda v_0) = \lambda^{2/(p+1)} X(\lambda^{(p-1)/(p+1)} t; v_0), \quad (2)$$

where  $X(t; v_0) = \{x_i(t; v_0), \text{ for all } i = 1, \dots, M\}$ . This corresponds to the so called-power self-similarity of the second kind, that is, the one that is defined from the dynamics [19]. Of course, this one-parameter transformation posses the group property  $T_{\lambda_1} \cdot T_{\lambda_2} = T_{\lambda_1 \lambda_2}$ , where  $T_{\lambda} X(t; v_0) = X(t; \lambda v_0)$ . There is also a corresponding equation for velocities.

The system is studied numerically by using an explicit Runge-Kutta method of 5th order based on the Dormand-Prince coefficients, with local extrapolation. As step size controller we have used the proportional-integral step control, which gives a smooth step size sequence.

In Figure 2 we show both, the solution for bead 28 corresponding to an impact velocity  $v_0 = 0.2$  m/s and the one with a scaled impact velocity  $\lambda v_0$ , with  $\lambda = 6$ .

It is worth to mention that Chatterjee [23], while studying a system of  $N$  identical particles with an inter-grain potential  $V \propto \delta^{p+1}$ , noticed that if a function  $\tilde{X}(t)$  satisfies the corresponding equations of motion, then so does the function  $\alpha^{2/(p-1)} \tilde{X}(\alpha t)$ , for any positive (arbitrarily chosen) number  $\alpha$ . It seems that he didn't noticed the self-similarity of solutions. There is of course a mapping that connect them: by choosing  $\alpha = \lambda^{(p-1)/(p+1)}$  and recognizing that  $\lambda$  corresponds to the scale factor of the initial impact velocity our result is recovered.

Also, it is important to stress that this self-similarity phenomenon is tacitly present in many works in the subject, starting from the works of Nesterenko [1], through the work of Hinch and Saint-Jean [5], till the work of Rosas and Lindenberg [11], where dimensionless variables where used.

As stressed by Shirkov [24], the symmetry found here is not a symmetry of the physical system or the equations of the problem at hand, but a symmetry of a solution considered as a function of the relevant physical variables and suitable initial conditions. This kind of symmetry can be related to the invariance of a physical quantity described by this solution with respect to the way in which the initial conditions are imposed.

To end up it is important to stress also that the self-similarity found in this work only appears in granular systems without pre-compression. In fact, if we allow the system to be pre-compressed (in this case we include a wall at the right side of the chain) the eqs. of motion read

$$m_i \frac{d^2 x_i}{dt^2} = k_1(l_0 - (x_i - x_{i-1}))^p - k_2(l_0 - (x_{i+1} - x_i))^p, \quad (3)$$

and for the same initial conditions we found the result shown in Fig. 3, for  $l_0 = 10^{-4}$ . It is clear then that self-similarity is lost when the system is loaded. Of course, the breaking of self-similarity is less noticeable for very low loading. This is related to the fact that, in dimensionless variables, the self-similarity transformation appears via scale transformations (with parameter  $v_0$ ) of time and space coordinates.

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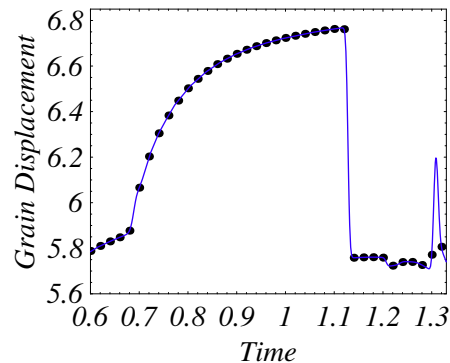


FIG. 2: Displacement of bead 28 as a function of time (in program units) for two different impact velocities  $v_0 = 0.2$  m/s (dots) and  $v_0 = 1.6$  m/s (full line).

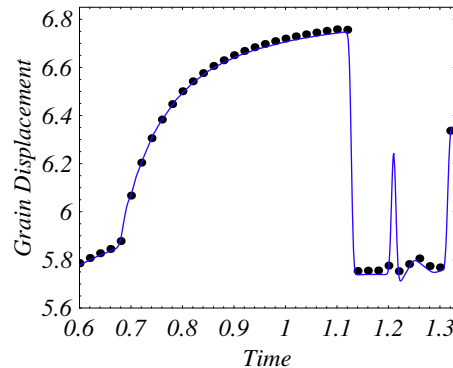


FIG. 3: Displacement of bead 28 as a function of time (in program units) for two different impact velocities  $v_0 = 0.2$  m/s (dots) and  $v_0 = 1.6$  m/s (full line), in case of a pre-compressed system.

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